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Identities in the Enveloping Algebras for Modular Lie Superalgebras

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INTRODUCTION

The universal enveloping algebra of a Lie algebra in characteristic zero satisfies a nontrivial identity if and only if the Lie algebra is abelian [13]. Necessary and sufficient conditions for a universal enveloping algebra in positive characteristic to satisfy a nontrivial identity have been found in [6]. In [8] the analogous problem for the universal enveloping algebra of a Lie superalgebra in characteristic zero has been settled. These results may also be found in the monograph [1]. The present author has also found necessary and sufficient conditions for the restricted envelope of a Lie p-algebra to be a PI-algebra [18, 19]. Independently, this result has been obtained by D. S. Passman using somewhat different methods [16, 17].

Our main results are Theorems 2.1 and 2.4 which give necessary and sufficient conditions for the restricted enveloping algebra of a restricted Lie superalgebra to satisfy a nontrivial identity. These theorems generalize previously mentioned results of [18]. Corollary 2.5 also generalizes a result of Ju. A. Bahturin [6].

In Section 6 the methods of this paper are used to specify those Lie superalgebras in both zero and positive characteristics such that the dimensions of all their irreducible representations are bounded by a finite constant (Theorems 6.2 and 6.3). The first of these results was obtained by Ju. A. Bahturin [10, 11].

From Theorem 2.1 we see that the properties of the restricted envelope of a Lie p-superalgebra resemble those of a group ring in positive characteristic. Indeed, necessary and sufficient conditions for a modular group ring to satisfy nontrivial identities [15] are close analogs of Theorem 2.1.

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1. Restricted Lie Superalgebras

The main field k is of positive characteristic $p>2$. In case $p=3$ we add to the axioms of a Lie superalgebra $L=L_0 \oplus L_1$ an identity $[[y, y], y] \equiv 0$, $y \in L_1$. (If $p>3$ then it follows from Jacobi's identity.) This is necessary to get an embedding of L into its universal enveloping algebra; see 1.2, 1.3.

We know from [14] the notion of a Lie p-superalgebra. Let us recall it supposing the notion of a Lie p-algebra is well known [2].

Definition 1.1. A Lie superalgebra $L=L_0 \oplus L_1$ over a field k of positive characteristic p is called restricted (or a p-superalgebra) if

(1) Lie algebra L_0 is restricted (i.e., has a unary operation

$$[p]: L_0 \rightarrow L_0; \qquad L_0 \ni X \mapsto X^{[p]} \in L_0$$

which satisfies some special conditions [2]).

(2) The action of p-algebra L_0 on module L_1 is restricted.

By $U(L)$ we shall denote, for a Lie superalgebra $L=L_0 \oplus L_1$, its universal enveloping algebra. (It exists if char $k \neq 2$ [1].)

Definition 1.2. Let us define for a Lie p-superalgebra over a field k with char $k=p \neq 2$ a restricted enveloping algebra by

$$u(L)=U(L)/\operatorname{Id}(x^p - x^{[p]} \mid x \in L_0).$$

Suppose that $L_0=\langle e_\alpha \mid \alpha \in I\rangle$, $L_1=\langle f_\beta \mid \beta \in J\rangle$ are ordered bases; then by analogy with [2] we have.

Lemma 1.3. *The restricted enveloping algebra has the standard basis*

$$u(L)=\langle e_{\alpha_1}^{n_1} \cdots e_{\alpha_m}^{n_m} f_{\beta_1} \cdots f_{\beta_t} \mid \alpha_1 < \cdots < \alpha_m, \beta_1 < \cdots < \beta_t, \qquad 0 \leqslant n_j \langle p\rangle.$$

The degree of a basis monomial v written above is the number $\deg(v)=n_1+\cdots+n_m+t$; *the degree of any element $w \in u(L)$ is maximum of degrees of basis monomials which enter its decomposition.*

Definition 1.4. For any $m=0, 1, 2, \ldots$, define

$$u_m=u_m(L)=\{w \in u(L) \mid \deg(w) \leqslant m\}.$$

By standard methods we have

LEMMA 1.5. (1) *The degree of an element defined above does not depend on the ordered basis of* L_0 *and* L_1.

(2) *The set* $\{u_m \mid m=0, 1, 2, ...,\}$ *forms a filtration. Its associated graded algebra is isomorphic to the tensor product* $\mathrm{gr}\{u_m\} \cong A_0 \otimes A_1$, *where* $A_0 = k[X_\alpha \mid \alpha \in I]/(X_\alpha^p \mid \alpha \in I)$ *is a ring of truncated polynomials, the number of variables being equal to the cardinality of* I; $A_1 = \Lambda(Y_\beta \mid \beta \in J)$ *is a Grassmann algebra, the number of variables being equal to the cardinality of* J.

If M is a homogeneous subalgebra in a Lie superalgebra $L = L_0 \oplus L_1$ then $M = M_0 \oplus M_1$ is the decomposition into its homogeneous components.

DEFINITION 1.6. Let M be a homogeneous subalgebra in a restricted Lie superalgebra L.

(1) Subalgebra M is called restricted if M_0 is a restricted Lie subalgebra in Lie p-algebra L_0.

(2) We call the minimal restricted subalgebra of L containing M the p-hull M_p of M.

Evidently, $M_p = (M_0)_p \oplus M_1$, where $(M_0)_p$ is the p-hull of the restricted Lie subalgebra $M_0 \subseteq L_0$.

LEMMA 1.7. *Let* L *be a restricted superalgebra and* $M \subseteq L$ *be a homogeneous restricted subalgebra of finite codimension. Then there exists a homogeneous restricted ideal* $\tilde{M} \subseteq L$ *of finite codimension in* L *which is contained in* M.

Proof. The same as in [12, Lemma 10] where this fact has been established for restricted Lie algebras. ∎

2. MAIN RESULT

THEOREM 2.1. *Let* $L = L_0 \oplus L_1$ *be a restricted Lie superalgebra over a field* k *of positive characteristic* $p \neq 2$. *Then the restricted enveloping algebra* $u(L)$ *satisfies a nontrivial identity if and only if there exist homogeneous restricted ideals* $Q \subseteq R \subseteq L$ *such that*

(1) $\dim L/R < \infty$, $\dim Q < \infty$.

(2) $R^2 \subseteq Q$, $Q^2 = 0$.

(3) *In* $Q = Q_0 \oplus Q_1$ *the restricted Lie subalgebra* $Q_0 \subseteq L_0$ *has a nilpotent p-map.*

Let us prove sufficiency conditions of the theorem without having supposed Q and R to be ideals.

PROPOSITION 2.2. *Suppose that there exist restricted homogeneous subalgebras $Q \subseteq R \subseteq L$ satisfying conditions* (1)–(3) *of Theorem* 2.1 *and that* $\dim L/R = t$, $\dim Q = m$. *Then $u(L)$ satisfies a nontrivial identity of degree* $108p^{2(m+t)}$.

Proof. Consider canonical embeddings $u(Q) \subseteq u(R) \subseteq u(L)$. The augmentation ideal $I = u_*(Q)$ *of* $u(Q)$ *is nilpotent*: $I^{\dim u(Q)} = 0$, where $\dim u(Q) = p^{\dim Q_0} 2^{\dim Q_1} \leqslant p^{\dim Q} = p^m$. The ideal J in $u(R)$ generated by Q is also nilpotent: $J = u(R)\, Q u(R) = u(R)\, Q$, $J^{p^m} = u(R)\, Q^{p^m} = 0$.

We have $u(R)/J = u(R/Q)$ (by analogy with [2]). By the hypothesis R/Q is a commutative superalgebra, hence $u(R/Q)$ satisfies an identity of the form $[[X, Y], Z] \equiv 0$. Therefore $u(R)$ satisfies $[[X, Y], Z]^{p^m} \equiv 0$.

By identifying any $x \in u(L)$ with a right multiplication we obtain an embedding of $u(L)$ into the endomorphism ring of a free module ${}_{u(R)}u(L)$ of finite rank $p^{\dim L_0/R_0} 2^{\dim L_1/R_1} \leqslant p^t$. Thus we have

$$u(L) \subseteq \operatorname{End}_{u(R)} u(L) \subseteq M_{p^t}(u(R)) \cong M_{p^t}(k) \otimes u(R).$$

By the proof of [1, 6.1.11], if algebras A, B satisfy nontrivial identities of degrees q_1, q_2, respectively, then, for $n! > (q_1 q_2)^{2n}$, there exists a nontrivial identity of degree n in $A \otimes B$.

Now the matrix ring $M_{p^t}(k)$ satisfies standard identity S_{2p^t}. Since $n! > (n/2)^n > (n/3)^n$, we obtain an inequality of the form $(n/3)^n > (6p^{m+t})^{2n}$. Hence by $n = 108p^{2(m+t)}$ the result follows. ∎

Remark 2.3. Let $Q \subseteq R \subseteq L$ be ideals satisfying the hypotheses of 2.2. Then $D = Qu(L)$ is an ideal in $u(L)$ with $D^{p^m} = 0$. By analogy, we obtain $u(L)/D \cong u(L/Q) \subseteq M_{p^t}(u(R/Q))$. Now $u(R/Q)$ satisfies $[X, Y]^2 \equiv 0$ and, by [20], the latter matrix ring satisfies a power of standard identity $(S_{2p^t})^N \equiv 0$ for $N = N(t) \in \mathbb{N}$. Thus $u(L)$ satisfies $(S_{2p^t})^{Np^m} \equiv 0$, the degree of this identity being $2Np^{m+t}$.

THEOREM 2.4. *Let L be a restricted superalgebra over a field k, with* $\operatorname{char} k = p > 2$. *Suppose that $u(L)$ satisfies a nontrivial identity of degree d. Then there exist homogeneous restricted subalgebras $Q \subseteq R \subseteq L$ such that*

(1) $\dim L/R < \infty$, $\dim Q < \infty$.

(2) $R^2 \subseteq Q$, $Q^2 = 0$.

(3) *The p-algebra Q_0 of $Q = Q_0 \oplus Q_1$ has nilpotent p-mapping.*

(4) *Moreover, if k is a perfect field then we have estimates of the form* $\dim L/R \leqslant 2^{42d^4}$, $\dim Q \leqslant 2^{27d^4}$.

Proof. Contained in Sections 3–5.

Proof. The proof of necessity of conditions of Theorem 2.1 immediately follows from 2.4. Indeed, by our assumption we have subalgebras $Q \subseteq R \subseteq L$ satisfying conditions (1)–(3) and we must construct a chain of ideals. By 1.7 we get an ideal $\tilde{R} \subseteq L$ with $\tilde{R} \subseteq R$, $\dim L/\tilde{R} < \infty$ and consider $\tilde{Q} = (\tilde{R}^2)_p \subseteq Q$. Then we get the chain $\tilde{Q} \subseteq \tilde{R} \subseteq L$, which completes the proof. ∎

From Theorem 2.1 we have:

COROLLARY 2.5. *Let $L = L_0 \oplus L_1$ be a superalgebra over field k with* char $k = p > 2$. *Then its universal enveloping algebra $U(L)$ satisfies a nontrivial identity if and only if there exist homogeneous subalgebras $B \subseteq A \subseteq L$ such that*

(1) $\dim L/A < \infty$, $\dim B < \infty$.

(2) $A^2 \subseteq B$.

(3) $B = B_1$; *that is,* $B_0 = 0$.

(4) *All inner derivations* ad x, $x \in L_0$ *defined on the whole of the superalgebra are algebraic and their degrees are bounded by some constant.*

Proof. Consider $U(L_0)^{(-)}$ as a Lie p-algebra with a natural p-mapping and take $(L_0)_p = L_0 + L_0^p + L_0^{p^2} + \cdots$, $L_p = (L_0)_p \oplus L_1$. Evidently $U(L) = u(L_p)$. Suppose that $U(L)$ is a *PI*-algebra. Then there exist restricted homogeneous ideals $Q \subseteq R \subseteq L_p$ satisfying (1)–(3) of Theorem 2.1. Note that $Q_0 = 0$, hence $A = R \cap L$, $B = Q \cap L$ are ideals as required.

Since $\dim L_p/R < \infty$, for each $x \in L_0$ there exists a p-polynomial f with $y = f(x) \in R$, so $(\operatorname{ad} y) L \subseteq R$, $(\operatorname{ad} y)^p L \subseteq R^2 \subseteq Q_1$, and among p-powers of an operator $(\operatorname{ad} y)^{p^2} = \operatorname{ad} f(x^{p^2})$ acting on a finite dimensional space Q_1 we find linearly dependent ones.

Let us prove the converse implication directly. By analogy with 2.2, $U(A)$ satisfies $[X, [Y, Z]]^N \equiv 0$, $N = 2^{\dim B}$. Choose bases $L_0 = \langle a_1, ..., a_n \rangle \oplus A_0$, $L_1 = \langle b_1, ..., b_m \rangle \oplus A_1$ with natural ordering. By assumption there exist p-polynomials $f_1, ..., f_n$ with $f_j(\operatorname{ad} a_j) = 0$ (because each polynomial divided some p-polynomial [2]). Thus $z_j = f_j(a_j)$ are central in $U(L)$. By analogy with [2, Chap. 5] $U(L)$ has the basis

$$a_1^{\alpha_1} \cdots a_n^{\alpha_n} \cdot b_1^{\beta_1} \cdots b_m^{\beta_m} \cdot z_1^{\gamma_1} \cdots z_n^{\gamma_n} \cdot w,$$
$$0 \leqslant \alpha_j < \deg(f_j),\ \beta_j \in \{0, 1\},\ \gamma_j = 0, 1, 2, ...,$$

where w are elements in the standard basis for $U(A)$. Subalgebra D generated by $U(A)$ and $z_1, ..., z_n$ satisfies all multilinear identities of $U(A)$.

Since ${}_DU(L)$ is a free module of finite rank, we obtain that $U(L)$ is a PI-algebra. ▮

Remarks. In this corollary we can substitute subalgebras for ideals; estimates in 2.4 remain valid. Moreover we do not need the restriction on the field to be perfect. (This follows by 5.1, since the p-mapping is transcendent in this case.)

Bounds on degree of polynomials in the corollary may be found also as in [1, 6.7.9; 6] by studying the action of ad x on L_1, $x \in L_0$.

All the results above also remain valid over a field of characteristic 2 if we restrict ourselves to the case of Lie p-algebras.

3. Two Identities

Let $A = A(X_1, ..., X_n, ..., Y_1, ..., Y_n, ...)$ be the free associative algebra in a countable number of variables. For any permutation $\pi \in S_n$ we define $f_\pi, f'_\pi \in A$ as

$$f_\pi = [X_1, Y_{\pi(1)}] \cdot \cdots \cdot [X_j, Y_{\pi(j)}] \cdot \cdots \cdot [X_n, Y_{\pi(n)}],$$
$$f'_\pi = (X_1 \circ Y_{\pi(1)}) \cdot \cdots \cdot (X_j \circ Y_{\pi(j)}) \cdot \cdots \cdot (X_n \circ Y_{\pi(n)}),$$

where $a \circ b = ab + ba$. Consider polynomials in A as

$$f(X_1, ..., X_n, Y_1, ..., Y_n) = \sum_{\pi \in S_n} \lambda_\pi \cdot f_\pi, \qquad \lambda_\pi \in k \tag{1}$$

$$f'(X_1, ..., X_n, Y_1, ..., Y_n) = \sum_{\pi \in S_n} \lambda'_\pi \cdot f'_\pi, \qquad \lambda'_\pi \in k. \tag{2}$$

Lemma 3.1. *If associative algebra B over some field satisfies a nontrivial identity of degree d then it also satisfies nontrivial identities of special type*

$$f(X_1, ..., X_n, Y_1, ..., Y_n) \equiv 0, \qquad f'(X_1, ..., X_n, Y_1, ..., Y_n) \equiv 0, \tag{3}$$

where $n = 3d^4$ and $f, f' \in A$ are of types (1) *and* (2), *respectively. Moreover we may take $\lambda_1 = 1$.*

Proof. Denote by $P_m(Z_1, ..., Z_m)$ the subspace of all associative multilinear polynomials in m variables $Z_1, ..., Z_m$ in free associative algebra $\tilde{A} = \tilde{A}(Z_1, ..., Z_i, ...)$ in a countable set of variables. By $T_m(Z_1, ..., Z_m)$ we denote the subspace of elements in $P_m(Z_1, ..., Z_m)$ which are identities for *PI-algebra B*. It is known that in this case we have an estimate of the form [1]

$$\dim P_m(Z_1, ..., Z_m)/T_m(Z_1, ..., Z_m) < d^{2m}, \qquad m \in \mathbb{N}. \tag{4}$$

Next we apply this inequality for A. In the subspace $P_{2n}(X_1, ..., X_n, Y_1, ..., Y_n) \subseteq A$ of multilinear polynomials of degree $2n$ depending on variables $X_1, ..., X_n, Y_1, ..., Y_n$ there exist $n!$ polynomials of type f_π, $\pi \in S_n$ which are linearly independent; the latter fact is clear from a form of a standard basis of a free associative algebra. By analogy we have $n!$ linearly independent polynomials f'_π, $\pi \in S_n$. Applying (4), we have

$$\dim P_{2n}(X_1, ..., X_n, Y_1, ..., Y_n)/T_{2n}(X_1, ..., X_n, Y_1, ..., Y_n) < d^{4n}, \qquad n \in \mathbb{N}.$$

If $n! > d^{4n}$ then we immediately get the desired identities (3). Since $n! > (n/e)^n > (n/3)^n$ the number $n = 3d^4$ is sufficiently great, and the result follows. ∎

Remark 3.2. Nontrivial elements of type (1) may be written in the form

$$\begin{aligned} f &= f(X_1 \cdots X_n, Y_1 \cdots Y_n) \\ &= \sum_{j=1}^{n} [X_1, Y_j] \sum_{\substack{\pi(1)=j \\ \pi \in S_n}} \lambda_\pi [X_2, Y_{\pi(2)}] \cdots [X_n, Y_{\pi(n)}] \\ &= \sum_{j=1}^{n} [X_1, Y_j] \cdot g_j, \qquad \text{where} \qquad (5) \\ g_j &= g_j(X_2 \cdots X_n, Y_1 \cdots \hat{Y}_j \cdots Y_n) \\ &= \sum_{\substack{\pi(1)=j \\ \pi \in S_n}} \lambda_\pi [X_2, Y_{\pi(2)}] \cdots [X_n, Y_{\pi(n)}]. \end{aligned}$$

The caret here denotes the argument which is omitted. Here at least one of g_j is nonzero (otherwise $f \equiv 0$), and all elements g_j are in fact of type (1); it suffices only to change indexes. All elements (2) have decomposition analogous to (5), where instead of commutators we write $(X \circ Y_j) = X_i Y_j + Y_j X_i$.

With the use of Lemma 3.1 the author avoided a lot of tedious computations in the study of restricted enveloping algebras for Lie p-algebras [18, 19] (see for comparison [16, 17]). It has applications in the study of irreducible representations; in particular, it allows the simplification of the proofs of [10, 11] (see Section 6). This lemma is of great importance in this paper, too. It may have some applications for the study of group rings; in particular for getting better bounds than in [15].

4. Delta Sets

Next we use a theorem of Neumann [1]:

THEOREM 4.1. *Let $\phi: U \times V \to W$ be a bilinear map, where U, V, W are vector spaces over field k. Suppose that for each $u \in U$ the codimension of its annihilator in V is bounded by m, and for each $v \in V$ the codimension of its annihilator in U is bounded by l. Then* $\dim \langle \phi(U, V) \rangle_k \leqslant ml$.

Consider Lie superalgebra $L = L_0 \oplus L_1$ over an arbitrary field k and introduce some useful sets, which for Lie algebras were introduced in [6].

DEFINITION 4.2. For Lie superalgebra $L = L_0 \oplus L_1$ define sets of four types for any $m = 0, 1, 2, ...$

$$\delta^m_{\alpha,\beta} = \delta^m_{\alpha,\beta}(L) = \{a \in L_\alpha \mid \dim [a, L_\beta] \leqslant m\}; \qquad \alpha, \beta = 0, 1.$$

Consider natural bilinear mapping $\phi: L_\alpha \times L_\beta \to L_{\alpha+\beta}$ which is induced by operation in the superalgebra (sums of indices in graduation are always considered modulo 2). Then $\delta^m_{\alpha,\beta}$ is the set of all $a \in L_\alpha$ such that the codimension of its annihilator in L_β is bounded by m. We shall denote Kroneker's delta-sign by $\delta_{i,j}$, which we hope does not create confusion.

DEFINITION 4.3. For any $m = 0, 1, 2, ...,$ define

$$\Delta^m_0 = \Delta^m_0(L) = \delta^m_{0,0} \cap \delta^m_{0,1}, \qquad \Delta^m_1 = \Delta^m_1(L) = \delta^m_{1,0} \cap \delta^m_{1,1}.$$

In hopes that there is no confusion, the argument in delta sets is omitted.

LEMMA 4.4. (1) *If* $x_1, ..., x_n \in \delta^m_{\alpha,\beta}$ $(\varepsilon \Delta^m_\alpha)$ *then* $\lambda_1 x_1 + \cdots + \lambda_n x_n \in \delta^{nm}_{\alpha,\beta}$ ($\varepsilon \Delta^{nm}_\alpha$, *respectively*) *for any* $\lambda_1, ..., \lambda_n \in k$.

(2) *If* $x \in \Delta^m_\alpha$, $y \in L_\gamma$ *then* $[x, y] \in \Delta^{2m}_{\alpha+\gamma}$.

(3) *If L is restricted and* $x \in \Delta^m_0$ *then* $x^{[p]} \in \Delta^m_0$.

(4) $\Delta^m_\alpha \subseteq \Delta^{m+1}_\alpha$.

Proof. Follows from inclusions

$$[\lambda_1 x_1 + \cdots + \lambda_n x_n, L_\beta] \subseteq [x_1, L_\beta] + \cdots + [x_n, L_\beta]$$
$$[[x, y], L_\beta] \subseteq [y, [x, L_\beta]] + [x, L_{\beta+\gamma}]$$
$$[x^{[p]}, L_\beta] \subseteq [x, ..., [x, L_\beta], ...,] \subseteq [x, L_\beta]. \quad \blacksquare$$

DEFINITION 4.5. $\Delta_0 = \bigcup_{j=0}^\infty \Delta^j_0$, $\Delta_1 = \bigcup_{j=0}^\infty \Delta^j_1$. That is, $\Delta = \Delta_0 \oplus \Delta_1$ consists of all elements of finite breadth.

COROLLARY 4.6. *$\Delta = \Delta_0 \oplus \Delta_1$ is a homogeneous ideal in Lie superalgebra L. This ideal is restricted if the superalgebra L is restricted.*

We now come to the proof of Theorem 2.4. From this moment it is supposed that the restricted enveloping algebra $u(L)$ for restricted Lie superalgebra $L = L_0 \oplus L_1$ satisfies a nontrivial identity of degree d. According to Lemma 3.1, we set $n = 3d^4$. Since $d \geqslant 2$, we have then $n \geqslant 48$.

THEOREM 4.7. *Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra over a field of prime characteristic $p \neq 2$. If the restricted enveloping algebra $u(L)$ satisfies a nontrivial identity of degree d, then any n elements in L_α are linearly dependent modulo $\delta^{n^2}_{\alpha\beta}$, this holding for any pair $\alpha, \beta \in \{0, 1\}$, where $n = 3d^4$.*

Proof. Fix α, β and set $\gamma = \alpha + \beta$. We shall use an identity of a second type in Lemma 3.1 if $\alpha = \beta = 1$ and one of the first type otherwise. The corresponding element of A will be denoted simply by f. Pick arbitrary $a_1, \ldots, a_n \in L_\alpha$ and substitute in $f \equiv 0$ the elements

$$Y_1 = a_1, \ldots, Y_n = a_n; \qquad X_1 = x_1, \ldots, X_n = x_n,$$

where $x_1, \ldots, x_n$ are arbitrary elements in L_β. Next in the proof if in some equality enter symbols $x_1, \ldots, x_n$ (or simply x) then it means that this equality is true for all $x_1, \ldots, x_n \in L_\beta$ ($x \in L_\beta$). In this proof brackets will denote only operations in superalgebra and via this substitution we have a formula fitting into both cases:

$$\begin{aligned} f(x_1 \cdots x_n, a_1 \cdots a_n) &= \sum_{\pi \in S_n} \lambda_\pi [x_1, a_{\pi(1)}] \cdot \cdots \cdot [x_n, a_{\pi(n)}] \\ &= \sum_{j=1}^{n} [x_1, a_j] \cdot g_j(x_2 \cdots x_n, a_1 \cdots \hat{a}_j \cdots a_n) \equiv 0, \\ &\qquad x_1, \ldots, x_n \in L_\beta. \end{aligned} \tag{6}$$

Note that the left-hand side is the sum of products, each product consisting of n factors in L; hence the left-hand side is always an element in u_n (see 1.4).

Since (6) holds it suffices to prove that the condition on any fixed elements $a'_1, \ldots, a'_m \in L_\alpha$,

$$f'(x_1, \ldots, x_m, a'_1, \ldots, a'_m) = 0 \pmod{u_{m-1}}, \qquad x_1, \ldots, x_m \in L_\beta,$$

where $0 \not\equiv f'(X_1 \cdots X_m, Y_1 \cdots Y_m) \in A$ is of types (1), (2) (respectively to fixed α, β), implies that $a'_1, \ldots, a'_m$ are linearly dependent modulo $\delta^{m^2}_{\alpha,\beta}$. Omitting for convenience the dashes and proceeding by induction on m we have the following.

If $m = 1$ then $f(x_1, a_1) = [x_1, a_1] \equiv 0$ for any $x_1 \in L_\beta$, hence $a_1 \in \delta^0_{\alpha,\beta}$.

Now suppose that our result holds for $m-1$, $m>1$. Without loss of generality we assume that $0 \not\equiv g_1(X_2 \cdots X_m, Y_2 \cdots Y_m) \in A$. There are two cases to consider. In the first case for any $x_2, ..., x_m \in L_\beta$ we have $g_1(x_2, ..., x_m, a_2, ..., a_m) = 0 \pmod{u_{m-2}}$, hence by induction hypothesis $a_2, ..., a_m$ are linearly dependent modulo $\delta_{\alpha,\beta}^{(m-1)^2} \subseteq \delta_{\alpha,\beta}^{m^2}$.

In the remaining case there exist $b_2, ..., b_m \in L_\beta$ such that $\deg(g_1(b_2, ..., b_m, a_2, ..., a_m)) = m-1$. In a fomula which is analogous to (6) we fix $x_2 = b_2, ..., x_m = b_m$;

$$\sum_{j=1}^{m} [x_1, a_j] \cdot g_j(b_2 \cdots b_m, a_1 \cdots \hat{a}_j \cdots a_m) \equiv 0 \pmod{u_{m-1}}, \qquad x_1 \in L_\beta.$$

If we write x in place of x_1, the summands where $g_j = g_j(b_2 \cdots b_m, a_1 \cdots \hat{a}_j \cdots a_m) \in u_{m-2}$ translate to the right-hand side; then we obtain

$$\sum_{j=1}^{r} [x, a_j] \cdot g_j \equiv 0 \pmod{u_{m-1}}\, x \in L_\beta, \qquad \deg(g_j) = m-1, j = 1, ..., r. \tag{7}$$

The next step is to prove, by induction, that this identity implies that $a_1, ..., a_r$ are linearly dependent modulo $\delta_{\alpha,\beta}^{m^2}$. Denote by V the subspace in L_γ, spanned by $[b_i, a_j]$, $2 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m$. Then $\dim V < m^2$ and $g_1, ..., g_r$ are in the subalgebra of $u(L)$ generated by V.

If $r=1$, then $[x, a_1] \cdot g_1 \equiv 0 \pmod{u_{m-1}}$ for all $x \in L_\beta$. Let us prove that this is possible only in the case where $[L_\beta, a_1] \subseteq V$; that is, if $a_1 \in \delta_{\alpha\beta}^{m^2}$. By way of contradiction, suppose $e = [b, a_1] \notin V$, $b \in L_\beta$. Choose an ordered basis for L whose first element is $e \in L_\gamma$ followed by a basis of $V = \langle v_1, ..., v_t \rangle \subseteq L_\gamma$. Now g_1 is the sum of products, each product consisting of $m-1$ factors of the form $[b_i, a_j] \subseteq V \subseteq L_\gamma$ which can be expressed as linear combinations of basis elements for V. Since $\deg(g_1) = m-1$, using the standard basis of the restricted enveloping algebra, we have

$$g_1 = \sum v_{j_1} \cdots v_{j_{m-1}} + v, \qquad \deg(v) < m-1. \tag{8}$$

Multiplying g_1 by $e = [b, a_1] \notin V$ on the left we obtain an element of degree m. Thus, with $x = b$, we have a contradiction since $[x, a_1] \cdot g_1 \equiv 0 \pmod{u_{m-1}}$ for all $x \in L_\beta$.

Consider $r>1$. If, in (7), $[L_\beta, a_r] \subseteq V$ holds, then $a_r \in \delta_{\alpha\beta}^{m^2}$ and the result follows. Thus we can assume $e = [b, a_r] \notin V$ for some $b \in L_\beta$. By analogy with the preceding argument we choose an ordered basis in L. We set $[b, a_j] = \alpha_j e + w_j$, $j = 1, ..., r-1$, $\alpha_j \in k$, where each w_j is a linear combination of basis elements for L_γ except e. By setting $x = b$ in (7) we obtain

$$e \cdot (\alpha_1 g_1 + \cdots + \alpha_{r-1} g_{r-1} + g_r) + w_1 \cdot g_1 + \cdots + w_{r-1} \cdot g_{r-1} \in u_{m-1}. \tag{9}$$

Denote $g=\alpha_1 g_1+\cdots+\alpha_{r-1} g_{r-1}+g_r$. Suppose that $\deg(g)=m-1$. By analogy with the preceding argument g is of the form (8); this means that the first summand in (9) has degree m and may be written so that

$$e\cdot g=\sum e\cdot v_{j_1}\cdots v_{j_{m-1}}+e\cdot v, \qquad \deg(e\cdot v)<m. \tag{10}$$

Other summands in (9) either have degree m or, being written in form (10), have basis elements of L distinct from e as their first factors. Since we can arbitrarily permute elements in (9) modulo right-hand side, we have a contradiction.

So $g=\alpha_1 g_1+\cdots+\alpha_{r-1} g_{r-1}+g_r\in u_{m-2}$. Take this expression for g and substitute it into (7):

$$\sum_{j=1}^{r-1}[x, a_j-\alpha_j a_r]\cdot g_j\equiv 0 \ (\mathrm{mod}\ u_{m-1}),\ x\in L_\beta,$$

$$\deg(g_j)=m-1, j=1, ..., r-1.$$

By inductive assumption $a_1-\alpha_1 a_r, ..., a_{r-1}-\alpha_{r-1} a_r$ are linearly dependent modulo $\delta^{m^2}_{\alpha\beta}$ and therefore the set $\{a_1, ..., a_r\}$ is linearly dependent. The theorem is proved. ▮

COROLLARY 4.8. *Under the hypothesis of the theorem there exist subspaces* $W_0\subseteq\Delta_0^{n^3}$, $W_1\subseteq\Delta_1^{n^3}$ *with* $\dim L_\alpha/W_\alpha\leqslant 2(n-1)$, $\alpha\in\{0, 1\}$.

Proof. By the theorem in order to get the linear span $\langle\delta^{n^2}_{\alpha,\beta}\rangle$ we can restrict ourselves to sums of n elements. Hence, by Lemma 4.4(1), $\langle\delta^{n^2}_{\alpha,\beta}\rangle\subseteq\delta^{n^3}_{\alpha,\beta}$. Also by the theorem, $\dim_3 L_\alpha/\langle\delta^{n^2}_{\alpha,\beta}\rangle\leqslant n-1$. Finally, we conclude that $W_\alpha=\langle\delta^{n^2}_{\alpha,0}\rangle\cap\langle\delta^{n^2}_{\alpha,1}\rangle\subseteq\Delta^{n^3}_\alpha$, $\alpha=0, 1$, are required subspaces. ▮

PROPOSITION 4.9. *There exists a restricted homogeneous subalgebra C such that*

(1) *C is nilpotent of rank* 2 *and its commutator subalgebra is of finite dimension*: $\dim C^2\leqslant n^8 2^{8n-4}$.

(2) $\dim L/C\leqslant n^{12}2^{12n-5}$.

Proof. Define sets $A^0=\Delta_0^{n^3}\cup\Delta_1^{n^3}$,

$$A^i=\{[x, y_1, ..., y_k]\mid x\in A^0, y_j\in\{L_0\cup L_1\}, 0\leqslant k\leqslant i\}, \qquad i\in\mathbb{N}.$$

These sets are unions of two nonintersecting subsets $A^i=A^i_0\cup A^i_1$, $i=0, 1, 2, ...,$ lying in the respective homogeneous components. Next we define

$$B^i_\alpha=\langle A^i_\alpha\rangle\subseteq L_\alpha, \qquad \alpha\in\{0, 1\}, \qquad B^i=B^i_0\oplus B^i_1\subseteq L.$$

By Corollary 4.8 $\dim L/B^0 \leqslant 4(n-1)$. We have a chain of subspaces which stabilizes due to the dimension argument:

$$B^0 \subseteq B^1 \subseteq B^2 \subseteq \cdots \subseteq B^t \subseteq B^{t+1} = \cdots \subseteq L.$$

Note that if $B^t = B^{t+1}$ then B^t is an ideal in L. Thus we obtain a restricted homogeneous ideal $H = B^t = B^{4(n-1)}$. Using 4.4(2) by induction we easily have

$$A_\alpha^{4(n-1)} \subseteq \Delta_\alpha^{N_1}, \qquad N_1 = n^3 2^{4(n-1)}, \qquad \alpha \in \{0, 1\}.$$

By Corollary 4.8 all linear spans $B_\alpha^i = \langle A_\alpha^i \rangle$, $i = 0, 1, \ldots$, may be considered to consist of sums with at most $2(n-1)$ summands. Now Lemma 4.4(1) yields

$$H_\alpha = \langle A_\alpha^{4(n-1)} \rangle \subseteq \Delta_\alpha^N, \qquad \text{where } N = n^4 2^{4n-3}, \qquad \alpha \in \{0, 1\}. \tag{11}$$

Applying Theorem 4.1 to the natural bilinear map $H_\alpha \times H_\beta \to H_{\alpha+\beta}$ we conclude that the dimension of each vector space $[H_0, H_0]$, $[H_0, H_1]$, $[H_1, H_1]$ is bounded by N^2, therefore

$$\dim H^2 \leqslant 3 \cdot N^2 \leqslant n^8 2^{8n-4} \tag{12}$$

We set $C = C_H(H^2) = \{x \in H \mid [x, H^2] = 0\}$, This is a homogeneous restricted subalgebra. It centralizes its commutator, hence it is nilpotent of rank 2. Thus Condition (1) follows from (12).

Note that $C = C_0 \oplus C_1$ is the intersection of centralizers in H of finitely many basis elements for H^2. By virtue of (11) we obtain

$$\dim H_\alpha / C_\alpha \leqslant N \cdot \dim H^2 \leqslant n^{12} 2^{12n-7}, \qquad \alpha \in \{0, 1\}.$$

Finally,

$$\dim L/C = \dim L/H + \dim H/C \leqslant 4n + n^{12} 2^{12n-6} \leqslant n^{12} 2^{12n-5},$$

completing the proof. ∎

Remark. In the proof of Theorem 4.7 we have not used that characteristic is positive. The proof holds for superalgebra $L = L_0 \oplus L_1$ in zero characteristic, too, such that the universal enveloping algebra $U(L)$ satisfies a nontrivial identity and we can get analogous conditions on four sets $\delta_{\alpha,\beta}^{n^2}(L) \subseteq L_\alpha$. It is enough for two sets in the odd component but not for two sets in the even one. Indeed, in such a superalgebra $\delta_{0,0}^0(L) = L_0$, $\delta_{0,1}^m(L) = L_0$ for some integer m [8].

5. The Case of 2-Step Nilpotent Superalgebra with Finite Dimensional Commutator Subalgebra

In Proposition 4.9 we have obtained a 2-step nilpotent superalgebra C with finite dimensional commutator subalgebra $C^2 = (C^2)_0 \oplus (C^2)_1$ and the proof of Theorem 2.4 is now, in fact, reduced to the study of this superalgebra. Subalgebra C^2 is not necessarily restricted and the dimension of its p-hull $(C^2)_p = ((C^2)_0)_p \oplus (C^2)_1$ may be infinite.

We start with some preliminary remarks on p-algebras. If we are concerned with some abelian p-algebra A then, by definition,

$$(\lambda_1 x_1 + \lambda_2 x_2)^{[p]} = \lambda_1^p x_1^{[p]} + \lambda_2^p x_2^{[p]}, \qquad x_1, x_2 \in A,\ \lambda_1, \lambda_2 \in k.$$

For convenience we write x^p instead of $x^{[p]}$. On the vector space $\Omega = \langle 1, t, t^2, t^3, ..., \rangle_k$ we introduce the multiplication

$$(\alpha t^i) \cdot (\beta t^j) = \alpha \beta^{p^i} \cdot t^{i+j}, \qquad \alpha, \beta \in k.$$

Then the restricted subalgebras in A are exactly submodules in the module ${}_\Omega A$, where the action of the ring Ω on A is determined by action of a generator $t \cdot x = x^{[p]}$, $x \in A$. The ring Ω, in the case of the perfect field, is a skew domain of principal ideals, hence for any finitely generated abelian Lie p-algebra A we have decomposition of ${}_\Omega A$ into the direct sum of finitely many cyclic submodules [4, Chap. 3]:

$${}_\Omega A = \langle s_1 \rangle_p \oplus \cdots \oplus \langle s_q \rangle_p, \tag{13}$$

where $\langle s_i \rangle_p = \langle s_i, s_i^p, ..., s_i^{p^j}, ..., \rangle$ is a p-hull of one dimensional Lie subalgebra $\langle s_i \rangle$.

Proposition 5.1. *Let C be as in* 4.9 *and the main field be perfect. Then there exists restricted homogeneous subalgebra G such that*

(1) *G is nilpotent of rank* 2, $\dim G^2 \leqslant n^8 2^{8n-4}$.

(2) $\dim (G^2)_p < \infty$.

(3) $\dim C/G \leqslant n^9 2^{8n-2}$.

Proof. (1) follows from 4.9.

Now $(C^2)_p$ is contained in the center of C. Since $\dim C^2 < \infty$, we have, by (13), that

$$((C^2)_0)_p = \langle s_1 \rangle_p \oplus \cdots \oplus \langle s_q \rangle_p = V. \tag{14}$$

A basis of $\langle s \rangle_p$ is formed by the sequence of all p-powes s^{p^j}, $j = 0, 1, 2, ...,$ or by its initial segment s^{p^j}, $0 \leqslant j \leqslant j_0$ (transcendental and algebraic cases, respectively). We shall prove that there exists a subalgebra $G \subseteq C$ without transcendent elements in the decomposition analogous to (14) and then condition (2) will be obvious.

Let s be a transcendent element in (14) and $\rho: (C^2)_0 \to k$ the projection on $\langle s \rangle_p$ followed by taking the coefficient of s in the above basis for $\langle s \rangle_p$. Consider the bilinear alternating form:

$$\phi: C_0 \times C_0 \to k, \qquad \phi(x, y) = \rho([x, y]), \qquad x, y \in C_0.$$

We claim that $\dim C_0/\mathrm{Ker}\ \phi < 2n$. Otherwise there exist $x_1, ..., x_n, y_1, ..., y_n \in C_0$ such that

$$\phi(x_i, x_j) = \phi(y_i, y_j) = 0, \qquad \phi(x_i, y_j) = \delta_{i,j}, \qquad 1 \leqslant i, j \leqslant n.$$

Substitute these into the first identity from Lemma 3.1:

$$\sum_{\pi \in S_n} \lambda_\pi [x_1, y_{\pi(1)}] \cdot \cdots \cdot [x_n, y_{\pi(n)}] \equiv 0 \tag{15}$$

Note that $u(\langle s \rangle_p) \cong k[s]$. If we choose a term in (15) corresponding to identity permutation, then we get s^n. Other terms from identity permutation and all terms from nonidentity permutations either contain at least one factor from $\langle s_j \rangle_p$, $s_j \neq s$ or have all n factors of form s^t, $t \geqslant 1$; furthermore, the latter nonequality in at least one case is strict $t > 1$. Therefore we have a contradiction with (15). Thus $\dim C_0/\mathrm{Ker}\ \phi < 2n$ holds.

By analogy, we construct a bilinear symmetric form

$$\psi: C_1 \times C_1 \to k, \qquad \psi(x, y) = \rho([x, y]), \qquad x, y \in C_1.$$

Let us prove that $\dim C_1/\mathrm{Ker}\ \psi < n$. Otherwise there exist $x_1, ..., x_n \in C_1$ with $\psi(x_i, x_j) = \delta_{i,j}$, $1 \leqslant i, j \leqslant n$. Substitute these into the second identity from Lemma 3.1:

$$\sum_{\pi \in S_n} \lambda_\pi (x_1 \circ x_{\pi(1)}) \cdot \cdots \cdot (x_n \circ x_{\pi(n)}) \equiv 0.$$

By analogy, we obtain a similar contradiction. Next we consider $\tilde{C} = \mathrm{Ker}\ \phi \oplus \mathrm{Ker}\ \psi$ with $\dim C/\tilde{C} \leqslant 3n$, $\rho((\tilde{C}^2)_0) = 0$. Note that by (14) the linear function ρ on $(C^2)_0$ is nontrivial, hence we have strict inclusion $\tilde{C}^2 \subset C^2$. Therefore, at most after $\dim C^2$ steps, we are able to get rid of transcendental elements in the decomposition analogous to (14) and we obtain a restricted homogeneous subalgebra G with

$$\dim C/G \leqslant 3n \cdot \dim C^2 \leqslant n^9 2^{8n-2},$$

and condition (3) is proved. ∎

DEFINITION 5.2. Let A be a finite dimensional abelian Lie p-algebra. Then $N(A)=\{x\in A \mid \exists 1 : x^p=0\}$ denotes the set of all nil-p-elements.

PROPOSITION 5.3. *Let G as in 5.1 and the main field be algebraically closed. Then there exists restricted homogeneous subalgebra $F\subseteq G$ such that*

(1) $((F^2)_0)_p$ *consists of nil-p-elements.*

(2) $\dim G/F\leqslant n^9 2^{8n-2}$.

Proof. We set, for convenience, $A=((G^2)_0)_p$. By [5] it has a basis

$$A=\langle e_1, ..., e_m \mid e_j^p=e_j\rangle_k \oplus N(A). \tag{16}$$

Consider $D=\langle e_2, ..., e_m\rangle_k \oplus N(A)$ and let $\rho: A\to k$ be a projection of A on $\langle e_1\rangle$ with kernel D. Consider the alternating form

$$\phi: G_0\times G_0\to k, \qquad \phi(x,y)=\rho([x,y]), \qquad x,y\in G_0.$$

Next we prove $\dim G_0/\operatorname{Ker}\phi<2n$. Otherwise there exists $x_1, ..., x_n$, $y_1, ..., y_n\in G_0$ with

$$\phi(x_1,x_j)=\phi(y_i,y_j)=0, \qquad \phi(x_i,y_j)=\delta_{i,j}, \qquad 1\leqslant i,j\leqslant n.$$

Substitute these into the first identity from Lemma 3.1:

$$\sum_{\pi\in S_n} \lambda_\pi [x_1, y_{\pi(1)}]\cdot \cdots \cdot [x_n, y_{\pi(n)}]\equiv 0 \tag{17}$$

By definition, $[x_i,y_j]=\delta_{i,j}e_1+z$, $1\leqslant i,j\leqslant n$, where $z\in D$. If we choose a term in (17) corresponding to identity permutation, then we get $0\neq e_1^n\in u(A)$. Other terms in (17) are contained in $u(A)\,u_*(D)$ (where $u_*(D)$ is an augmentation ideal in $u(D)$).

Hence (17) yields a contradiction, so $\dim G_0/\operatorname{Ker}\phi<2n$. By analogy with 5.1, if we consider the mapping

$$\psi: G_1\times G_1\to k, \qquad \psi(x,y)=\rho([x,y]), \qquad x,y\in G_1$$

then, using the second identity from Lemma 3.1, we obtain $\dim G_1/\operatorname{Ker}\psi<n$. If $\tilde{G}=\operatorname{Ker}\phi\oplus\operatorname{Ker}\psi$ then, easily, $\dim G/\tilde{G}\leqslant 3n$, $\rho((\tilde{G}^2)_0)=0$. Note that $\rho((G^2)_0)\not\equiv 0$ (otherwise $A=((G^2)_0)_p\subseteq D$). Therefore the inclusion $\tilde{G}^2\subset G^2$ is strict and after at most $\dim G^2$ steps we obtain a subalgebra F, satisfying condition (1) with

$$\dim G/F\leqslant \dim G^2\cdot 3n\leqslant n^9 2^{8n-2}$$

and the result follows. Finally we remark that F_0, F_1 are the kernels of bilinear mappings

$$\Phi: G_0 \times G_0 \to A/N(A), \qquad \Psi: G_1 \times G_1 \to A/N(A). \quad \blacksquare$$

PROPOSITION 5.4. *Let G as in 5.1. Then there exist restricted homogeneous subalgebras $Q \subseteq R \subseteq G$ such that*

(1) $R^2 \subseteq Q$, $Q^2 = 0$.

(2) $Q = Q_0 \oplus Q_1$ *is finite dimensional and Lie p-algebra Q_0 has nilpotent p-mapping*, $\dim G/R < \infty$.

(3) *If k is perfect then*

$$\dim G/R \leqslant n^9 2^{8n-1}, \qquad \dim Q \leqslant n^9 2^{8n-4}.$$

Proof. Let $\bar{k}$ be an algebraic closure of the perfect field k and, given a subalgebra H, we denote $\bar{H} = \bar{k} \otimes_k H$. Then it is easy to prove that for any finite dimensional abelian Lie p-algebra A an equality $N(\bar{A}) = \overline{N(A)}$ holds [18]; in particular this is true for $A = ((G^2)_0)_p$.

We consider bilinear mappings

$$\Phi: G_0 \times G_0 \to A/N(A), \qquad \Phi': \bar{G}_0 \times \bar{G}_0 \to \bar{A}/N(\bar{A}) = \overline{A/N(A)}.$$

Let Ψ, Ψ' be defined by analogy. Then $\operatorname{Ker} \Phi' = \overline{\operatorname{Ker} \Phi}$ and, for $F = \operatorname{Ker} \Phi \oplus \operatorname{Ker} \Psi$, by the final remark in 5.3, we obtain $\dim G/F \leqslant n^9 2^{8n-2}$ and $(F^2)_0 \subseteq N(A)$. By (13), for $B = ((F^2)_0)_p$,

$$B = \langle s_1 \rangle_p \oplus \cdots \oplus \langle s_m \rangle_p, \tag{18}$$

where all s_i, $i = 1, ..., m$ are nulpotent. Let s be one of these. Assume $s^{p^n} \neq 0$. As in 5.1 we define $\rho: B \to k$ and

$$\phi: F_0 \times F_0 \to k, \qquad \phi(x, y) = \rho([x, y]), \qquad x, y \in F_0.$$

We claim $\dim F_0/\operatorname{Ker} \phi < 2n$. Otherwise there exist $x_1, ..., x_n, y_1, ..., y_n \in F_0$ with $\phi(x_i, x_j) = \phi(y_i, y_j) = 0$, $\phi(x_i, y_j) = \delta_{i,j}$, $1 \leqslant i, j \leqslant n$. Substitute these into the first identity from Lemma 3.1:

$$\sum_{\pi \in S_n} \lambda_\pi [x_1, y_{\pi(1)}] \cdot \cdots \cdot [x_n, y_{\pi(n)}] \equiv 0.$$

If we choose a term corresponding to identity permutation, then we get $s^n \neq 0$. Other terms either have factors in $\langle s_j \rangle_p$, $s_j \neq s$ or have all n factors of the form s^t, $t \geqslant 1$, nonequality at least one time being strict. This is contradiction. Next for $\psi: F_1 \times F_1 \to k$ we prove $\dim F_1/\operatorname{Ker} \psi < n$. By analogy with 5.1 for subalgebra $\tilde{F} = \operatorname{Ker} \phi \oplus \operatorname{Ker} \psi$ we have $\dim F/\tilde{F} < 3n$ and $\dim \tilde{F}^2 < \dim F^2$.

At most after $\dim F^2$ steps we get a subalgebra R having decomposition analogous to (18) where for all s we have $s^{p^n} = 0$, that is, $\dim \langle s \rangle_p \leqslant n$. Hence R and $Q = (R^2)_p$ satisfy required conditions:

$$\dim G/R \leqslant \dim G/F + \dim F/R \leqslant n^9 2^{8n-1},$$

$$\dim Q \leqslant \dim R^2 \cdot n \leqslant \dim G^2 \cdot n \leqslant n^9 2^{8n-4}.$$

Consider now the case where k is an arbitrary field. Note that, in order to get decompositions (13) and (16), it suffices to extend the field by adjoining finitely many roots of p-polynomials [2]. Let $k \subseteq K$ be such an extension with basis $K = \langle f_1, ..., f_m \rangle_k$, $f_1 = 1$. Let $\bar{\ }$ be as above. Consider

$$\Phi : G_0 \times G_0 \to A/N(A), \qquad \Phi' : \bar{G}_0 \times \bar{G}_0 \to \bar{A}/N(\bar{A}).$$

By a similar argument to the one above $\dim_K \bar{G}_0/\operatorname{Ker} \Phi' = l$. Pick up arbitrary $x_1, ..., x_t \in G_0$, $t > 1$. Consider the vector spaces

$$U = \{(\mu_1, ..., \mu_t) \mid \mu_i \in K\}, \qquad U \supseteq W = \{(\lambda_1, ..., \lambda_t) \mid \lambda_i \in k\}, \qquad \dim_k W = t,$$
$$U \supseteq V = \{(\mu_1, ..., \mu_t) \mid \mu_i \in K, \mu_1 x_1 + \cdots + \mu_t x_t \in \operatorname{Ker} \Phi'\},$$
$$\dim_k V \geqslant m(t - l).$$

As soon as $\dim_k V + \dim_k W > \dim_k U$, that is, if $t > ml$, there exist $\lambda_1, ..., \lambda_t \in k$ with $\lambda_1 x_1 + \cdots + \lambda_t x_t \in \operatorname{Ker} \Phi' \cap G_0 \subseteq \operatorname{Ker} \Phi$. So $\dim G_0/\operatorname{Ker} \Phi \leqslant ml$. By analogy, we construct Ψ, $\operatorname{Ker} \Psi$. Finally $R = \operatorname{Ker} \Phi \oplus \operatorname{Ker} \Psi$, $Q = (R^2)_p$ are subalgebras as required. ∎

We are now ready to complete the proof.

Proof of Theorem 2.4. Propositions 4.9, 5.1, 5.4 yield a chain of restricted homogeneous subalgebras $L \supseteq C \supseteq G \supseteq R \supseteq Q$. Remark that subalgebras R, Q satisfy conditions (1)–(3). It remains only to check the bounds when k is perfect. Recall that $n = 3d^4 \geqslant 48$. We have

$$\begin{aligned} \dim L/R &= \dim L/C + \dim C/G + \dim G/R \leqslant n^{12} 2^{12n-5} + n^9 2^{8n-2} + n^9 2^{8n-1} \\ &\leqslant n^{12} 2^{12n-5} + n^9 2^{8n} \leqslant n^{12} 2^{12n-4}, \end{aligned}$$
$$\dim Q \leqslant n^9 2^{8n-4}.$$

Now we need one simple fact.

LEMMA 5.5. *For any integer* $n \geqslant 48$ *one has* (1) $n^9 \leqslant 16 \cdot 2^n$, (2) $n^3 \leqslant 2^n$.

Proof of Lemma. The first inequality for $n=48$ is verified directly. If we pass from n to $n+1$ then the value of the right-hand side doubles whereas the value of the left-hand side multiplies at most by $(\frac{49}{48})^9 \leqslant 1.3$. ▮

Finally we have

$$\dim L/R \leqslant 2^{14n} = 2^{42d^4}, \qquad \dim Q \leqslant 2^{9n} = 2^{27d^4},$$

and the proof is complete. ▮

6. Irreducible Representations of Lie Superalgebras

We know the following result:

THEOREM 6.1 [7]. *Let L be a Lie algebra over an algebraically closed field k. Then the degrees of all irreducible L-representations are bounded by some finite number if and only if* $\dim L < \operatorname{card} k$ *and one of the following conditions hold*:

(1) $\operatorname{char} k = 0$, *$L$ abelian.*

(2) $\operatorname{char} k > 0$, *$L$ contains an abelian ideal H of finite codimension and all derivations* $\operatorname{ad} x$, $x \in L$ *are algebraic of bounded degree.*

By $J(A)$ we denote the Jacobson radical for an associative algebra A.

As it is proved in [10], for $\mathbb{Z}_2$-graded associative algebra $A = A_0 \oplus A_1$ with 1 the intersection of annihilators of all graded irreducible modules coincides with $J(A)$. Moreover, a condition for the degrees of all irreducible representations to be bounded by some finite number is equivalent to an analogous condition on graded irreducible representations. Therefore two theorems below can be viewed in two ways.

The goal of this section is to prove, by means of the above technique, Theorems 6.2 and 6.3, the first of which was proved in [10, 11].

THEOREM 6.2 [11]. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra over an algebraically closed field k,* $\operatorname{char} k = 0$. *Then the degrees of all irreducible L-representations are bounded by some finite number if and only if*

(1) *L_0 abelian.*

(2) $\dim_k L_0 < \operatorname{card} k$.

(3) *There exists an L_0-submodule of finite codimension M in L_1 with*

(4) $[M, M] = 0$.

THEOREM 6.3. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra over an algebraically closed field k of prime characteristic $p > 2$. Then the degrees of all irreducible L-representations are bounded by some finite number if and only if there exists homogeneous restricted ideal $R = R_0 \oplus R_1 \subseteq L$ with*

(1) $\dim L/R < \infty$.

(2) $\dim_k R_0 < \operatorname{card} k$.

(3) $R^2 \subseteq R_1$.

(4) *All derivations* $\operatorname{ad} x|_{L_0}$, $x \in L_0$ *are algebraic of bounded degree.*

Note that the main and essential difference between conditions of 2.5 and 6.3 is that in the latter case we do not have any statements about L_0-module L_1, and in particular we do not need $R^2 = [R_0, R_1]$ to be finite dimensional (latter equality follows from condition (3) of the theorem).

LEMMA 6.4. *Let A be a subalgebra with 1 in an associative algebra B with the same unity, and B_A be a free module.*

(1) *If degrees of all irreducible B-modules are bounded by some number, then the same number bounds the degrees of all irreducible A-modules, too.*

(2) *If we have decomposition $B_A = \bigoplus_{j \in J} v_j A \oplus 1A$, then $J(B) \cap A \subseteq J(A)$.*

Proof. (1), in fact, proved in [1, 6.8.4; 7]. For (2) assume that $r \in J(B) \cap A$; then $1 - rs$ has an inverse for any $s \in A$. By the given decomposition we easily derive that this inverse element belongs to A. ∎

Proof of Theorems 6.2, 6.3. Suppose that first we are given $R \subseteq L$ satisfying (1)–(4) from 6.3. Choose a basis $L_0 = \langle a_1, ..., a_m \rangle \oplus R_0$ and p-polynomials $f_j(X)$, $j = 1, ..., m$, with $f_j(\operatorname{ad} a_j)|_{L_0} = 0$; then $z_j = f_j(a_j)$ are central in $U(L_0)$. Consider the commutative subalgebra $D_0 \subseteq U(L_0)$ generated by $z_1, ..., z_m$ with $U(R_0)$ and a subalgebra D generated by D_0 and R_1. By [10] $J(D) = DR_1$. From 6.1 we know that the degrees of all irreducible representations for $D_0 \cong D/J(D)$ and hence for D are bounded by a finite number (moreover, they all are one dimensional).

Now it remains to show that $U(L)$ is a free module of finite rank over D and to apply arguments as in [9, 10] (sufficiency of conditions of Theorem 6.2; see [10]).

For the converse, assume that degrees of all irreducible $U(L)$-representations are bounded by some constant and the characteristic is arbitrary. Then $U(L)/J(U(L))$ can be embedded into the direct sum of matrix rings over k of bounded order. Therefore $U(L)$ satisfies some nontrivial polynomial identity modulo its Jacobson radical. Let us choose a second identity from Lemma 3.1 and pick up arbitrary $y_1, ..., y_n, z_1, ..., z_n \in L_1$. By 6.4(2) and the fact that $J(U(L_0)) = 0$ (see [3]) we obtain

$$\sum_{\pi \in S_n} \lambda_\pi (y \circ z_{\pi(1)}) \cdot \cdots \cdot (y_n \circ z_{\pi(n)}) = 0. \tag{19}$$

Applying literally the proof of Theorem 4.7 we conclude that any $y_1, ..., y_n \in L_1$ are linearly dependent modulo $\delta_{1,1}^{n^2}$. Analogously to 4.9 by consideration of sets $F^0 = \delta_{1,1}^{n^2}$,

$$F^i = \langle [x, y_1, ..., y_1] \mid x \in \delta_{1,1}^{n^2}, y_j \in L_0, 0 \leqslant 1 \leqslant i \rangle, \qquad i \in \mathbb{N},$$

we get L_0-submodule $M \subseteq L_1$, with $\dim L_1/M < n$ and $M \subseteq \delta_{1,1}^N(L)$ for some integer N. Then by P. M. Neumann's Theorem 4.1 we have $\dim [M, M] \leqslant N^2$. Let $[M, M] = \langle v_1, ..., v_m \rangle = V$ be the basis and $\rho_s: V \to k$ be the operator of taking the coeficient of v_s. Consider natural mappings $\Psi: M \times M \to [M, M]$ and

$$\psi_s: M \times M \to k, \qquad \psi_s(x, y) = \rho_s([x, y]), \qquad x, y \in M, 1 \leqslant s \leqslant m.$$

Show that $\dim M/\mathrm{Ker}\, \psi_s < n$ for any $1 \leqslant s \leqslant m$. Otherwise there exist $y_1, ..., y_n \in M$ with $\psi_s(y_1, y_j) = \delta_{i,j}$, $1 \leqslant i, j \leqslant n$, and these we substitute in (19)

$$\sum_{\pi \in S_n} \lambda_\pi (y_1 \circ y_{\pi(1)}) \cdot \cdots \cdot (y_n \circ y_{\pi(n)}) = 0.$$

Identity permutation gives as one of its terms v_s^n which cannot cancel with other terms. Therefore $\dim M/\mathrm{Ker}\, \Psi < nm$.

Note that $L_0 \oplus \mathrm{Ker}\, \Psi$ is an ideal in L. Finally it remains to apply 6.1 to Lie algebra L_0 because of 6.4(1). ▌

Finally note that analogous problems on irreducible representations for restricted enveloping algebras and modular group rings remain open.

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